Branching Law for the Finite Subgroups of SL(4,C)

Frédéric Butin

Branching Law for the Finite Subgroups of $\mathbf{SL}_4\mathbb{C}$

Frédéric BUTIN[1]

Abstract
In the framework of McKay correspondence we determine, for every finite subgroup Γ of $\mathbf{SL}_4\mathbb{C}$, how the finite dimensional irreducible representations of $\mathbf{SL}_4\mathbb{C}$ decompose under the action of Γ.
Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_4\mathbb{C}$ and let ϖ_1, ϖ_2, ϖ_3 be the corresponding fundamental weights. For $(p, q, r) \in \mathbb{N}^3$, the restriction $\pi_{p,q,r}|_\Gamma$ of the irreducible representation $\pi_{p,q,r}$ of highest weight $p\varpi_1 + q\varpi_2 + r\varpi_3$ of $\mathbf{SL}_4\mathbb{C}$ decomposes as $\pi_{p,q,r}|_\Gamma = \bigoplus_{i=0}^{l} m_i(p, q, r)\gamma_i$. We determine the multiplicities $m_i(p, q, r)$ and prove that the series $P_\Gamma(t, u, w)_i = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} m_i(p, q, r)t^p u^q w^r$ are rational functions.
This generalizes results from Kostant for $\mathbf{SL}_2\mathbb{C}$ and our preceding works about $\mathbf{SL}_3\mathbb{C}$.

Keywords: McKay correspondence; branching law; representations; finite subgroups of $\mathbf{SL}_4\mathbb{C}$.

Mathematics Subject Classifications (2000): 20C15; 17B10; 15A09; 17B67.

1 Introduction and results

• Let Γ be a finite subgroup of $\mathbf{SL}_4\mathbb{C}$ and $\{\gamma_0, \ldots, \gamma_l\}$ the set of equivalence classes of irreducible finite dimensional complex representations of Γ, where γ_0 is the trivial representation. The character associated to γ_j is denoted by χ_j.
Consider $\gamma : \Gamma \to \mathbf{SL}_4\mathbb{C}$ the natural $4-$dimensional representation, and γ^* its contragredient representation. The character of γ is denoted by χ. By complete reducibility we get the decompositions

$$\forall\ j \in [\![0,\, l]\!],\ \ \gamma_j \otimes \gamma = \bigoplus_{i=0}^{l} a_{ij}^{(1)}\gamma_i\ \ ,\ \ \gamma_j \otimes (\gamma \wedge \gamma) = \bigoplus_{i=0}^{l} a_{ij}^{(2)}\gamma_i\ \text{ and }\ \gamma_j \otimes \gamma^* = \bigoplus_{i=0}^{l} a_{ij}^{(3)}\gamma_i.$$

This defines the three following square matrices of $\mathbf{M}_{l+1}\mathbb{N}$:

$$A^{(1)} := \left(a_{ij}^{(1)}\right)_{(i,j)\in[\![0,\, l]\!]^2},\ A^{(2)} := \left(a_{ij}^{(2)}\right)_{(i,j)\in[\![0,\, l]\!]^2} \text{ and } A^{(3)} := \left(a_{ij}^{(3)}\right)_{(i,j)\in[\![0,\, l]\!]^2}.$$

• Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_4\mathbb{C}$ and let $\varpi_1,\ \varpi_2,\ \varpi_3$ be the corresponding fundamental weights, and $V(p\varpi_1 + q\varpi_2 + r\varpi_3)$ the simple $\mathfrak{sl}_4\mathbb{C}-$module of highest weight $p\varpi_1 + q\varpi_2 + r\varpi_3$ with $(p,\, q,\, r) \in \mathbb{N}^3$. Then we get an irreducible representation $\pi_{p,q,r} : \mathbf{SL}_4\mathbb{C} \to \mathbf{GL}(V(p\varpi_1 + q\varpi_2 + r\varpi_3))$. The restriction of $\pi_{p,q,r}$ to the subgroup Γ is a representation of Γ, and by complete reducibility, we get the decomposition

$$\pi_{p,q,r}|_\Gamma = \bigoplus_{i=0}^{l} m_i(p, q, r)\gamma_i,$$

where the $m_i(p, q, r)$'s are non negative integers. Let $\mathcal{E} := (e_0, \ldots, e_l)$ be the canonical basis of $\mathbb{C}^{l+1}$, and

$$v_{p,q,r} := \sum_{i=0}^{l} m_i(p, q, r)e_i \in \mathbb{C}^{l+1}.$$

We have in particular $v_{0,0,0} = e_0$ as γ_0 is the trivial representation. Let us consider the vector

$$P_\Gamma(t,\, u,\, w) := \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} v_{p,q,r}t^p u^q w^r \in (\mathbb{C}[\![t,\, u,\, w]\!])^{l+1},$$

[1]Université de Lyon, Université Lyon 1, CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918, F-69622 Villeurbanne-Cedex, France, email: butin@math.univ-lyon1.fr

and denote by $P_\Gamma(t,\,u,\,w)_j$ its $j-$th coordinate in the basis $\mathcal{E}$, which is an element of $\mathbb{C}[\![t,\,u,\,w]\!]$. Note that $P_\Gamma(t,\,u,\,w)$ can also be seen as a formal power series with coefficients in $\mathbb{C}^{l+1}$. The aim of this article is to prove the following theorem.

Theorem 1
The coefficients of $P_\Gamma(t,\,u,\,w)$ are rational fractions in $t,\,u,\,w$, i.e. the formal power series $P_\Gamma(t,\,u,\,w)_i$ are rational functions

$$P_\Gamma(t,\,u,\,w)_i = \frac{N_\Gamma(t,\,u,\,w)_i}{D_\Gamma(t,\,u,\,w)},\ i \in [\![0,\,l]\!],$$

where the $N_\Gamma(t,\,u,\,w)_i$'s and $D_\Gamma(t,\,u,\,w)$ are elements of $\mathbb{Q}[t,\,u,\,w]$.

• The proof of this theorem uses a key-relation satisfied by $P_\Gamma(t,\,u,\,w)$ as well as a so-called inversion formula. Two essential ingredients are the decomposition of the tensor product of $\pi_{p,q,r}$ with the natural representation of $\mathbf{SL}_4\mathbb{C}$ and the simultaneous diagonalizability of certain matrices. The effective calculation of $P_\Gamma(t,\,u,\,w)$ then reduces to matrix multiplication.

In [BP09] we applied a similar method for $\mathbf{SL}_2\mathbb{C}$ — recovering thereby in a quite easy way the results obtained by Kostant in [Kos85], [Kos06], and by Gonzalez-Sprinberg and Verdier in [GSV83] — and for $\mathbf{SL}_3\mathbb{C}$ in order to get explicit computations of the series for every finite subgroup of $\mathbf{SL}_3\mathbb{C}$.

The general framework of that study is the construction of a minimal resolution of singularities of the orbifold $\mathbb{C}^n/\Gamma$. It is related to the McKay correspondence (see [BKR01], [GSV83] and [GNS04]). For example, Gonzalez-Spriberg and Verdier use in [GSV83] a Poincaré series to construct explicitly minimal resolutions for singularities of $V = \mathbb{C}^2/\Gamma$ when Γ is a finite subgroup of $\mathbf{SL}_2\mathbb{C}$. To go further in this approach, our results for $\mathbf{SL}_4\mathbb{C}$ could be used to construct an explicit synthetic minimal resolution of singularities for orbifolds of the form $\mathbb{C}^4/\Gamma$ where Γ is a finite subgroup of $\mathbf{SL}_4\mathbb{C}$.

2 Properties of the matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$

In order to compute the series $P_\Gamma(t,\,u,\,w)$, we first establish here some properties of the matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$. The first proposition essentially follows from the uniqueness of the decomposition of a representation as sum of irreducible representations.

Proposition 2
• $A^{(3)} = {}^tA^{(1)}$.
• $A^{(2)}$ *is a symmetric matrix.*
• $A^{(1)}$, $A^{(2)}$ *and* $A^{(3)}$ *commute. In particular,* $A^{(1)}$ *is a normal matrix.*

Proof:
Since $a^{(1)}_{ij} = (\chi_i \,|\, \chi_{\gamma\otimes\gamma_j}) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\overline{\chi_i(g)}\chi(g)\chi_j(g)$, we have $\gamma\otimes\gamma_j = \bigoplus_{i=0}^l a^{(1)}_{ij}\gamma_i$. In the same way, $(\gamma\wedge\gamma)\otimes\gamma_j = \bigoplus_{i=0}^l a^{(2)}_{ij}\gamma_i$ and $\gamma^*\otimes\gamma_j = \bigoplus_{i=0}^l a^{(3)}_{ij}\gamma_i$.
Then

$$\begin{array}{rcl} a^{(3)}_{ij} & = & (\chi_i \,|\, \chi_{\gamma_j\otimes\gamma^*}) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\overline{\chi_i(g)}\chi_j(g)\chi_{\gamma^*}(g) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\overline{\chi_i(g)}\chi_j(g)\chi(g^{-1}) \\ & = & \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\overline{\chi_i(g^{-1})}\chi_j(g^{-1})\chi(g) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_i(g)\overline{\chi_j(g)}\chi(g) = a^{(1)}_{ji}, \end{array}$$

hence $A^{(3)} = {}^tA^{(1)}$.
We also have $(\gamma_j\otimes\gamma)\otimes\gamma^* = \left(\bigoplus_{i=0}^l a^{(1)}_{ij}\gamma_i\right)\otimes\gamma^* = \bigoplus_{i=0}^l a^{(1)}_{ij}\left(\bigoplus_{k=0}^l a^{(3)}_{ki}\gamma_k\right) = \bigoplus_{k=0}^l\left(\sum_{i=0}^l a^{(3)}_{ki}a^{(1)}_{ij}\right)\gamma_k$ and $\gamma\otimes(\gamma_j\otimes\gamma^*) = \gamma\otimes\left(\bigoplus_{i=0}^l a^{(3)}_{ij}\gamma_i\right) = \bigoplus_{i=0}^l a^{(3)}_{ij}\left(\bigoplus_{k=0}^l a^{(1)}_{ki}\gamma_k\right) = \bigoplus_{k=0}^l\left(\sum_{i=0}^l a^{(1)}_{ki}a^{(3)}_{ij}\right)\gamma_k$, hence $A^{(3)}A^{(1)} = A^{(1)}A^{(3)}$. The proofs of the other statements are the same. ■

Since $A^{(1)}$, $A^{(2)}$, $A^{(3)}$ are normal, we know that they are diagonalizable with eigenvectors forming an orthogonal basis. Now we will diagonalize these matrices by using the character table of the group Γ. Let us denote by $\{C_0, \ldots, C_l\}$ the set of conjugacy classes of Γ, and for any $j \in [\![0, l]\!]$, let g_j be an element of C_j. So the character table of Γ is the matrix $T_\Gamma \in \mathbf{M}_{l+1}\mathbb{C}$ defined by $(T_\Gamma)_{i,j} := \chi_i(g_j)$.

Proposition 3
• *For $k \in [\![0, l]\!]$, set $w_k := (\chi_0(g_k), \ldots, \chi_l(g_k)) \in \mathbb{C}^{l+1}$. Then w_k is an eigenvector of $A^{(3)}$ associated to the eigenvalue $\chi(g_k)$. Similarly, w_k is an eigenvector of $A^{(1)}$ associated to the eigenvalue $\overline{\chi(g_k)}$.*
• *For $k \in [\![0, l]\!]$, w_k is an eigenvector of $A^{(2)}$ associated to the eigenvalue $\frac{1}{2}\left(\chi(g_k)^2 + \chi(g_k^2)\right)$.*

Proof:
From the relation $\gamma_i \otimes \gamma = \sum_{j=0}^{l} a_{ji}^{(1)} \gamma_j$, we get $\chi_i \chi = \chi_{\gamma_i \otimes \gamma} = \sum_{j=0}^{l} a_{ji}^{(1)} \chi_j$. By evaluating this on g_k, we obtain $\chi_i(g_k)\chi(g_k) = \sum_{j=0}^{l} a_{ji}^{(1)} \chi_j(g_k) = \sum_{j=0}^{l} a_{ij}^{(3)} \chi_j(g_k)$ according to Proposition 2. So w_k is an eigenvector of $A^{(3)}$ associated to the eigenvalue $\chi(g_k)$. The method is similar for the other results. ■

As the w_j's are the column of T_Γ, which are always orthogonal, the matrix T_Γ is invertible and the family $\mathcal{W} := (w_0, \ldots, w_l)$ is a common basis of eigenvectors of $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$. Then $\Lambda^{(1)} := T_\Gamma^{-1} A^{(1)} T_\Gamma$, $\Lambda^{(2)} := T_\Gamma^{-1} A^{(2)} T_\Gamma$ and $\Lambda^{(3)} := T_\Gamma^{-1} A^{(3)} T_\Gamma$ are diagonal matrices, with $\Lambda_{jj}^{(1)} = \overline{\chi(g_j)}$, $\Lambda_{jj}^{(2)} = \frac{1}{2}(\chi(g_j)^2 - \chi(g_j^2))$ and $\Lambda_{jj}^{(3)} = \chi(g_j)$.

Now, we make use of the Clebsch-Gordan formula

$$\begin{array}{rcl} \pi_{1,0,0} \otimes \pi_{p,q,r} & = & \pi_{p+1,q,r} \oplus \pi_{p,q,r-1} \oplus \pi_{p-1,q+1,r} \oplus \pi_{p,q-1,r+1}, \\ \pi_{0,1,0} \otimes \pi_{p,q,r} & = & \pi_{p,q+1,r} \oplus \pi_{p,q-1,r} \oplus \pi_{p+1,q-1,r+1} \oplus \pi_{p-1,q+1,r-1} \oplus \pi_{p-1,q,r+1} \oplus \pi_{p+1,q,r-1}, \\ \pi_{0,0,1} \otimes \pi_{p,q,r} & = & \pi_{p,q,r+1} \oplus \pi_{p-1,q,r} \oplus \pi_{p,q+1,r-1} \oplus \pi_{p+1,q-1,r}. \end{array} \tag{1}$$

Proposition 4
The vectors $v_{m,n}$ satisfy the following recurrence relations

$$\begin{array}{rcl} A^{(1)} v_{p,q,r} & = & v_{p+1,q,r} + v_{p,q,r-1} + v_{p-1,q+1,r} + v_{p,q-1,r+1}, \\ A^{(2)} v_{p,q,r} & = & v_{p,q+1,r} + v_{p,q-1,r} + v_{p+1,q-1,r+1} + v_{p-1,q+1,r-1} + v_{p-1,q,r+1} + v_{p+1,q,r-1}, \\ A^{(3)} v_{p,q,r} & = & v_{p,q,r+1} + v_{p-1,q,r} + v_{p,q+1,r-1} + v_{p+1,q-1,r}. \end{array}$$

Proof:
The definition of $v_{p,q,r}$ reads $v_{p,q,r} = \sum_{i=0}^{l} m_i(p, q, r) e_i$, thus $A^{(1)} v_{p,q,r} = \sum_{i=0}^{l} \left(\sum_{j=0}^{l} m_j(p, q, r) a_{ij}^{(1)} \right) e_i$. Now

$$(\pi_{1,0,0} \otimes \pi_{p,q,r})|_\Gamma = \pi_{p,q,r}|_\Gamma \otimes \gamma = \sum_{j=0}^{l} m_j(p, q, r) \gamma_j \otimes \gamma = \sum_{i=0}^{l} \left(\sum_{j=0}^{l} m_j(p, q, r) a_{ij}^{(1)} \right) \gamma_i,$$

and

$$\begin{aligned} & \pi_{p+1,q,r}|_\Gamma + \pi_{p,q,r-1}|_\Gamma + \pi_{p-1,q+1,r}|_\Gamma + \pi_{p,q-1,r+1}|_\Gamma \\ & \qquad = \sum_{i=0}^{l} \left(m_i(p+1, q, r) + m_i(p, q, r-1) + m_i(p-1, q+1, r) + m_i(p, q-1, r+1) \right) \gamma_i. \end{aligned}$$

By uniqueness,

$$\sum_{j=0}^{l} m_j(p, q, r) a_{ij}^{(1)} = m_i(p+1, q, r) + m_i(p, q, r-1) + m_i(p-1, q+1, r) + m_i(p, q-1, r+1). \ ■$$

3 The series $P_\Gamma(t, u, w)$ is a rational function

This section is mainly devoted to the proof of Theorem 1.

3.1 A key-relation satisfied by the series $P_\Gamma(t,\,u,\,w)$

Proposition 5
Set

$$\begin{aligned} J(t,\,u,\,w) \;&:=\; (1-u^2)((1+ut^2)(1+uw^2)-tw(1+u^2))I_n + twu(1-u^2)A^{(2)} \\ &\quad -tu(1+uw^2)(A^{(3)}-uA^{(1)}) - wu(1+ut^2)(A^{(1)}-uA^{(3)}). \end{aligned}$$

Then the series $P_\Gamma(t,\,u,\,w)$ satisfies the following relation

$$\begin{aligned} J(t,\,u,\,w)\,v_{0,0,0} \;&=\; \left(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4\right)\left(1-wA^{(3)}+w^2A^{(2)}-w^3A^{(1)}+w^4\right) \\ &\quad \left((1+u^2)(1-u^2)^2-u(1-u^2)^2A^{(2)}+u^2(A^{(1)}-uA^{(3)})(A^{(3)}-uA^{(1)})\right)P_\Gamma(t,\,u,\,w). \end{aligned}$$

Proof:

• Set $x := P_\Gamma(t,\,u,\,w)$. Set also $v_{p,q,-1} := 0$, $v_{p,-1,r} := 0$ and $v_{-1,q,r} := 0$ for $(p,\,q,\,r) \in \mathbb{N}^3$, such that, according to the Clebsch-Gordan formula, the formulae of the preceding corollary are still true for $(p,\,q,\,r) \in \mathbb{N}^3$. So we have (by denoting $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}$ by $\sum_{pqr}$)

$$\begin{aligned} &(1-wA^{(3)}+w^2A^{(2)}-w^3A^{(1)}+w^4)x \\ =\;& \sum_{pqr} v_{p,q,r}t^pu^qw^r - \sum_{pqr}(v_{p,q,r+1}+v_{p-1,q,r}+v_{p,q+1,r-1}+v_{p+1,q-1,r})t^pu^qw^{r+1} \\ &+\sum_{pqr}(v_{p,q+1,r}+v_{p,q-1,r}+v_{p+1,q-1,r+1}+v_{p-1,q+1,r-1}+v_{p-1,q,r+1}+v_{p+1,q,r-1})t^pu^qw^{r+2} \\ &-\sum_{pqr}(v_{p+1,q,r}+v_{p,q,r-1}+v_{p-1,q+1,r}+v_{p,q-1,r+1})t^pu^qw^{r+3}+\sum_{pqr} v_{p,q,r}t^pu^qw^{r+4}, \end{aligned}$$

hence

$$(1-wA^{(3)}+w^2A^{(2)}-w^3A^{(1)}+w^4)x = (1-tw+uw^2-t^{-1}uw)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} v_{p,q,0}t^pu^q + t^{-1}uw\sum_{q=0}^{\infty} v_{0,q,0}u^q. \quad (2)$$

• In the same way (by denoting $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}$ by $\sum_{pq}$)

$$\begin{aligned} &(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} v_{p,q,0}t^pu^q \\ =\;& \sum_{pq} v_{p,q,0}t^pu^q - \sum_{pq}(v_{p+1,q,0}+v_{p-1,q+1,0}+v_{p,q-1,1})t^{p+1}u^q \\ &+\sum_{pq}(v_{p,q+1,0}+v_{p,q-1,0}+v_{p+1,q-1,1}+v_{p-1,q,1})t^{p+2}u^q \\ &-\sum_{pq}(v_{p,q,1}+v_{p-1,q,0}+v_{p+1,q-1,0})t^{p+3}u^q+\sum_{pq} v_{p,q,0}t^{p+4}u^q, \end{aligned}$$

hence

$$(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} v_{p,q,0}t^pu^q = (1+t^2u)\sum_{q=0}^{\infty} v_{0,q,0}u^q - tu\sum_{q=0}^{\infty} v_{0,q,1}u^q. \qquad (3)$$

Moreover, we have

$$\begin{aligned} &(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)\sum_{q=0}^{\infty} v_{0,q,0}u^q \\ =\;& \sum_{q=0}^{\infty} v_{0,q,0}u^q - \sum_{q}^{\infty}(v_{1,q,0}+v_{0,q-1,1})tu^q + \sum_{q}^{\infty}(v_{0,q+1,0}+v_{0,q-1,0}+v_{1,q-1,1})t^2u^q \\ &-\sum_{q}^{\infty}(v_{0,q,1}+v_{1,q-1,0})t^3u^q + \sum_{q}^{\infty} v_{0,q,0}t^4u^q, \end{aligned}$$

hence

$$
\begin{aligned}
&(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)\sum_{p=0}^{\infty}v_{0,q,0}u^q \\
= \;& (1+t^4+t^2u^{-1}+t^2u)\sum_{q=0}^{\infty}v_{0,q,0}u^q - t^2u^{-1}v_{0,0,0}-(t+t^3u)\sum_{q=0}^{\infty}v_{1,q,0}u^q \\
&-(tu+t^3)\sum_{q=0}^{\infty}v_{0,q,1}u^q + t^2u\sum_{q=0}^{\infty}v_{1,q,1}u^q.
\end{aligned} \tag{4}
$$

By combining Equations (2), (3) and (4), we get

$$
\begin{aligned}
&(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)(1-wA^{(3)}+w^2A^{(2)}-w^3A^{(3)}+w^4)x \\
= \;& (1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)\left((1-tw+uw^2-t^{-1}uw)\sum_{pq}v_{p,q,0}t^pu^q + t^{-1}uw\sum_{q=0}^{\infty}v_{0,q,0}u^q\right) \\
= \;& (1-tw+uw^2-t^{-1}uw)\left((1+t^2u)\sum_{q=0}^{\infty}v_{0,q,0}u^q - tu\sum_{q=0}^{\infty}v_{0,q,1}u^q\right) \\
&+(1+t^4+t^2u^{-1}+t^2u)t^{-1}uw\sum_{q=0}^{\infty}v_{0,q,0}u^q - twv_{0,0,0}-(1+t^2u)uw\sum_{q=0}^{\infty}v_{1,q,0}u^q \\
&-(u+t^2)uw\sum_{q=0}^{\infty}v_{0,q,1}u^q + tu^2w\sum_{q=0}^{\infty}v_{1,q,1}u^q,
\end{aligned}
$$

hence

$$
\begin{aligned}
&(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)(1-wA^{(3)}+w^2A^{(2)}-w^3A^{(1)}+w^4)x \\
= \;& (1+ut^2)(1+uw^2)\sum_{q=0}^{\infty}v_{0,q,0}u^q - tu(1+uw^2)\sum_{q=0}^{\infty}v_{0,q,1}u^q \\
&-wu(1+ut^2)\sum_{q=0}^{\infty}v_{1,q,0}u^q - twv_{0,0,0}+tu^2w\sum_{q=0}^{\infty}v_{1,q,1}u^q.
\end{aligned} \tag{5}
$$

Besides, we have the two following equations

$$
A^{(1)}\sum_{q=0}^{\infty}v_{0,q,0}u^q = \sum_{q=0}^{\infty}v_{1,q,0}u^q + u\sum_{q=0}^{\infty}v_{0,q,1}u^q, \tag{6}
$$

and

$$
A^{(3)}\sum_{q=0}^{\infty}v_{0,q,0}u^q = \sum_{q=0}^{\infty}v_{0,q,1}u^q + u\sum_{q=0}^{\infty}v_{1,q,0}u^q. \tag{7}
$$

From these two equations, we deduce

$$
\sum_{q=0}^{\infty}v_{0,q,1}u^q = (1-u^2)^{-1}(A^{(3)}-uA^{(1)})\sum_{q=0}^{\infty}v_{0,q,0}u^q. \tag{8}
$$

Now, we have

$$
A^{(1)}\sum_{q=0}^{\infty}v_{0,q,1}u^q = \sum_{q=0}^{\infty}v_{1,q,1}u^q + \sum_{q=0}^{\infty}v_{0,q,0}u^q + u\sum_{q=0}^{\infty}v_{0,q,2}u^q, \tag{9}
$$

and

$$
A^{(3)}\sum_{q=0}^{\infty}v_{0,q,1}u^q = \sum_{q=0}^{\infty}v_{0,q,2}u^q + u^{-1}\sum_{q=0}^{\infty}v_{0,q,0}u^q + u\sum_{q=0}^{\infty}v_{1,q,1}u^q - u^{-1}v_{0,0,0}, \tag{10}
$$

therefore

$$\sum_{q=0}^{\infty} v_{1,q,1}u^q = (1-u^2)^{-1}(A^{(1)}-uA^{(3)})\sum_{q=0}^{\infty} v_{0,q,1}u^q - (1-u^2)^{-1}v_{0,0,0}.$$

So, according to Equation (8), we deduce

$$\sum_{q=0}^{\infty} v_{1,q,1}u^q = (1-u^2)^{-2}(A^{(1)}-uA^{(3)})(A^{(3)}-uA^{(1)})\sum_{q=0}^{\infty} v_{0,q,0}u^q - (1-u^2)^{-1}v_{0,0,0}. \tag{11}$$

By using Equation (11), we may write Equation (5) as

$$\begin{aligned}
&(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)(1-wA^{(3)}+w^2A^{(2)}-w^3A^{(1)}+w^4)x \\
= \;& \Big((1+ut^2)(1+uw^2)+tu^2w(1-u^2)^{-2}(A^{(1)}-uA^{(3)})(A^{(3)}-uA^{(1)})\Big)\sum_{q=0}^{\infty} v_{0,q,0}u^q \\
&-tu(1+uw^2)\sum_{q=0}^{\infty} v_{0,q,1}u^q - wu(1+ut^2)\sum_{q=0}^{\infty} v_{1,q,0}u^q - (tw+tu^2w(1-u^2)^{-1})v_{0,0,0}.
\end{aligned} \tag{12}$$

From Equations (6) and (7), we also deduce

$$\sum_{q=0}^{\infty} v_{1,q,0}u^q = (1-u^2)^{-1}(A^{(1)}-uA^{(3)})\sum_{q=0}^{\infty} v_{0,q,0}u^q. \tag{13}$$

So, by using Equations (8) and (13), we obtain

$$\begin{aligned}
&(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)(1-wA^{(3)}+w^2A^{(2)}-w^3A^{(1)}+w^4)x \\
= \;& \Big((1+ut^2)(1+uw^2)-tu(1+uw^2)(1-u^2)^{-1}(A^{(3)}-uA^{(1)}) \\
&-wu(1+ut^2)(1-u^2)^{-1}(A^{(1)}-uA^{(3)})+tu^2w(1-u^2)^{-2}(A^{(1)}-uA^{(3)})(A^{(3)}-uA^{(1)})\Big)\sum_{q=0}^{\infty} v_{0,q,0}u^q \\
&-(tw+tu^2w(1-u^2)^{-1})v_{0,0,0},
\end{aligned} \tag{14}$$

i.e., by multiplying (14) by $(1-u^2)^2$,

$$\begin{aligned}
&(1-u^2)^2(1-tA^{(1)}+t^2A^{(2)}-t^3A^{(3)}+t^4)(1-wA^{(3)}+w^2A^{(2)}-w^3A^{(1)}+w^4)x \\
= \;& \Big((1-u^2)^2(1+ut^2)(1+uw^2)-tu(1+uw^2)(1-u^2)(A^{(3)}-uA^{(1)}) \\
&-wu(1+ut^2)(1-u^2)(A^{(1)}-uA^{(3)})+tu^2w(A^{(1)}-uA^{(3)})(A^{(3)}-uA^{(1)})\Big)\sum_{q=0}^{\infty} v_{0,q,0}u^q \\
&-(tw(1-u^2)^2+tu^2w(1-u^2))v_{0,0,0}.
\end{aligned} \tag{15}$$

• Consider now the following equation

$$A^{(2)}\sum_{q=0}^{\infty} v_{0,q,0}u^q = u^{-1}\sum_{q=0}^{\infty} v_{0,q,0}u^q + u\sum_{q=0}^{\infty} v_{0,q,0}u^q + u\sum_{q=0}^{\infty} v_{1,q,1}u^q - u^{-1}v_{0,0,0}. \tag{16}$$

Then, according to Equation (11), we have

$$\begin{aligned}
A^{(2)}\sum_{q=0}^{\infty} v_{0,q,0}u^q \;=\;& u^{-1}\sum_{q=0}^{\infty} v_{0,q,0}u^q + u\sum_{q=0}^{\infty} v_{0,q,0}u^q \\
&+u(1-u^2)^{-2}(A^{(1)}-uA^{(3)})(A^{(3)}-uA^{(1)})\sum_{q=0}^{\infty} v_{0,q,0}u^q - u(1-u^2)^{-1}v_{0,0,0} - u^{-1}v_{0,0,0},
\end{aligned}$$

i.e.

$$\Big(A^{(2)} - u^{-1} - u - u(1-u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\Big) \sum_{q=0}^{\infty} v_{0,q,0} u^q = -(u(1-u^2)^{-1} + u^{-1}) v_{0,0,0}. \tag{17}$$

This last equation reads

$$\Big(-u(1-u^2)^2 A^{(2)} + (1+u^2)(1-u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\Big) \sum_{q=0}^{\infty} v_{0,q,0} u^q = (1-u^2) v_{0,0,0}. \tag{18}$$

Now, by using Equations (15) and (18), we get

$$\begin{array}{rl} & (1-u^2)^2(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4) \\ & \Big(-u(1-u^2)^2A^{(2)} + (1+u^2)(1-u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\Big)\, x \\ = & -tw(1-u^2)\Big(-u(1-u^2)^2A^{(2)} + (1+u^2)(1-u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\Big)\, v_{0,0,0} \\ & \Big((1-u^2)^2(1+ut^2)(1+uw^2) - tu(1+uw^2)(1-u^2)(A^{(3)} - uA^{(1)}) \\ & -wu(1+ut^2)(1-u^2)(A^{(1)} - uA^{(3)}) + tu^2w(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\Big)(1-u^2)v_{0,0,0}, \end{array} \tag{19}$$

i.e., after simplification by $(1-u^2)$,

$$\begin{array}{rl} & (1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4) \\ & \Big((1+u^2)(1-u^2)^2 - u(1-u^2)^2A^{(2)} + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\Big)\, x \\ = & \Big((1-u^2)((1+ut^2)(1+uw^2) - tw(1+u^2)) + twu(1-u^2)A^{(2)} \\ & -tu(1+uw^2)(A^{(3)} - uA^{(1)}) - wu(1+ut^2)(A^{(1)} - uA^{(3)})\Big) v_{0,0,0}. \end{array} \tag{20}$$

The proposition is proved. ■

3.2 An inversion formula

In order to inverse the relation obtained in Proposition 5 and get an explicit expression for $P_\Gamma(t,\, u)$, we need the rational function f defined by

$$\begin{array}{rcll} f & : & \mathbb{C}^3 & \to \ \mathbb{C}(t,\, u,\, w) \\ & & (d_1,\, d_2,\, d_3) & \mapsto \ (1 - td_1 + t^2d_2 - t^3d_3 + t^4)^{-1}(1 - wd_3 + w^2d_2 - w^3d_1 + w^4)^{-1} \\ & & & \qquad \Big((1+u^2)(1-u^2)^2 - u(1-u^2)^2d_2 + u^2(d_1 - ud_3)(d_3 - ud_1)\Big)^{-1}. \end{array}$$

According to Proposition 5, we may write

$$\begin{array}{rcl} J(t,\, u,\, w)\, v_{0,0,0} & = & T_\Gamma \left(1 - t\Lambda^{(1)} + t^2\Lambda^{(2)} - t^3\Lambda^{(3)} + t^4\right) \left(1 - w\Lambda^{(3)} + w^2\Lambda^{(2)} - w^3\Lambda^{(1)} + w^4\right) \\ & & \Big((1+u^2)(1-u^2)^2 - u(1-ur)^2\Lambda^{(2)} + u^2(\Lambda^{(1)} - u\Lambda^{(3)})(\Lambda^{(3)} - u\Lambda^{(1)})\Big) T_\Gamma^{-1} P_\Gamma(t,\, u,\, w). \end{array}$$

We deduce that

$$P_\Gamma(t,\, u,\, w) = T_\Gamma\, \Delta(t,\, u,\, w)\, T_\Gamma^{-1} J(t,\, u,\, w)\, v_{0,0,0} = (T_\Gamma\, \Delta(t,\, u,\, w)\, T_\Gamma)\, (T_\Gamma^{-2} J(t,\, u,\, w)\, v_{0,0,0}), \tag{21}$$

where $\Delta(t,\, u,\, w) \in \mathbf{M}_{l+1}\mathbb{C}(t,\, u,\, w)$ is the diagonal matrix defined by

$$\Delta(t,\, u,\, w)_{jj} = f(\Lambda_{jj}^{(1)},\, \Lambda_{jj}^{(2)},\, \Lambda_{jj}^{(3)}) = f\left(\overline{\chi(g_j)},\, \frac{1}{2}(\chi(g_j)^2 - \chi(g_j^2)),\, \chi(g_j)\right).$$

This last formula proves Theorem 1.

Remark 6
The Poincaré series $\widehat{P}_\Gamma(t)$ *of the algebra of invariants* $\mathbb{C}[z_1,\, z_2,\, z_3,\, z_4]^\Gamma$ *is given by*

$$\widehat{P}_\Gamma(t) = P_\Gamma(t,\, 0,\, 0)_0 = P_\Gamma(0,\, 0,\, t)_0.$$

3.3 Remark for $\mathbf{SL}_n\mathbb{C}$

In this section, we consider an integer $n \geq 2$ and a subgroup Γ of $\mathbf{SL}_n\mathbb{C}$. As in paragraph 1, let $\{\gamma_0, \ldots,\, \gamma_l\}$ be the set of equivalence classes of irreducible finite dimensional complex representations of Γ, where γ_0 is the trivial representation. The character associated to γ_j is denoted by χ_j.
Consider $\gamma : \Gamma \to \mathbf{SL}_n\mathbb{C}$ the natural $n-$dimensional representation, and χ its character. By complete reducibility we get the decomposition $\gamma_j \otimes \gamma = \bigoplus_{i=0}^{l} a_{ij}^{(1)}\gamma_i$ for every $j \in [\![0,\, l]\!]$, and we set $A^{(1)} := \left(a_{ij}^{(1)}\right)_{(i,j)\in[\![0,\, l]\!]^2} \in \mathbf{M}_{l+1}\mathbb{N}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_n\mathbb{C}$ and let $\varpi_1, \ldots,\, \varpi_{n-1}$ be the corresponding fundamental weights, and $V(p_1\varpi_1 + \cdots + p_{n-1}\varpi_{n-1})$ the simple $\mathfrak{sl}_n\mathbb{C}-$module of highest weight $p_1\varpi_1 + \cdots + p_{n-1}\varpi_{n-1}$ with $\mathbf{p} := (p_1, \ldots p_{n-1}) \in \mathbb{N}^{n-1}$. Then we get an irreducible representation $\pi_{\mathbf{p}} : \mathbf{SL}_n\mathbb{C} \to \mathbf{GL}(V(p_1\varpi_1 + \cdots + p_{n-1}\varpi_{n-1}))$. The restriction of $\pi_{\mathbf{p}}$ to the subgroup Γ is a representation of Γ, and by complete reducibility, we get the decomposition $\pi_{\mathbf{p}}|_\Gamma = \bigoplus_{i=0}^{l} m_i(\mathbf{p})\gamma_i$, where the $m_i(\mathbf{p})$'s are non negative integers. Let $\mathcal{E} := (e_0, \ldots,\, e_l)$ be the canonical basis of $\mathbb{C}^{l+1}$, and

$$v_{\mathbf{p}} := \sum_{i=0}^{l} m_i(\mathbf{p})e_i \in \mathbb{C}^{l+1}.$$

As γ_0 is the trivial representation, we have $v_{\mathbf{0}} = e_0$. Let us consider the vector (with elements of $\mathbb{C}[\![t_1, \ldots,\, t_{n-1}]\!] = \mathbb{C}[\![\mathbf{t}]\!]$ as coefficients)

$$P_\Gamma(\mathbf{t}) := \sum_{\mathbf{p}\in\mathbb{N}^{n-1}} v_{\mathbf{p}}\mathbf{t}^{\mathbf{p}} \in (\mathbb{C}[\![\mathbf{t}]\!])^{l+1},$$

and denote by $P_\Gamma(\mathbf{t})_j$ its $j-$th coordinate in the basis $\mathcal{E}$.

Given the results from Kostant ([Kos85] and [Kos06]) for $\mathbf{SL}_2\mathbb{C}$ and our results ([BP09]) about $\mathbf{SL}_3\mathbb{C}$, we then formulate the following conjecture:

Conjecture 7
The coefficients of the vector $P_\Gamma(\mathbf{t})$ *are rational fractions in* $\mathbf{t}$*, i.e. the formal power series* $P_\Gamma(\mathbf{t})_i$ *are rational functions*

$$P_\Gamma(\mathbf{t})_i := \frac{N_\Gamma(\mathbf{t})_i}{D_\Gamma(\mathbf{t})},\ i \in [\![0,\, l]\!],$$

where the $N_\Gamma(\mathbf{t})_i$*'s and* $D_\Gamma(\mathbf{t})$ *are elements of* $\mathbb{Q}[\mathbf{t}]$.

4 An example of explicit computation

The classification of finite subgroups of $\mathbf{SL}_4\mathbb{C}$ is given in [HH01]. It consists in infinite series and 30 exceptional groups (types $I,\, II, \ldots,\, XXX$). We give here an explicit computation of $P_\Gamma(t,\, u,\, w)$ for one of these exceptional groups. Consider the matrices

$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & j^2 \end{pmatrix},\ F_2' = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix},\ F_3' = \frac{1}{4}\begin{pmatrix} -1 & \sqrt{15} & 0 & 0 \\ \sqrt{15} & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix},$$

and the subgroup $\Gamma = \langle F_1,\ F_2',\ F_3' \rangle$ of $\mathbf{SL}_4\mathbb{C}$ (type II in [HH01]).
Here $l = 4$,

$$A^{(1)} = A^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad A^{(2)} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

$\mathrm{rank}(A^{(1)}) = \mathrm{rank}(A^{(2)}) = 4$, and the eigenvalues of $A^{(1)},\ A^{(2)},\ A^{(3)}$ are
$\Theta^{(1)} = \Theta^{(3)} = (4,\ 0,\ -1,\ 1,\ -1)$, $\Theta^{(2)} = (6,\ -2,\ 1,\ 0,\ 1)$, $p = 4$, and $\tau_0 = s_0 s_1$, $\tau_1 = s_2$, $\tau_2 = s_3$, $\tau_3 = s_4$.

According to formula 21, we get

$$\begin{aligned} D_\Gamma(t,\, u,\, w) &= (w-1)^4\,(u+1)^3\,(u-1)^5\,(t-1)^4\,\left(w^2+w+1\right)\left(w^4+w^3+w^2+w+1\right) \\ &\quad (w+1)^2\left(u^4+u^3+u^2+u+1\right)\left(u^2+u+1\right)^2\left(t^2+t+1\right)\left(t^4+t^3+t^2+t+1\right)(t+1)^2 \\ &= (u-1)(u+1)(u^2+u+1)\widetilde{D}_\Gamma(t)\widetilde{D}_\Gamma(u)\widetilde{D}_\Gamma(w), \end{aligned}$$

with $\widetilde{D}_\Gamma(t) = (t-1)^4(t+1)^2(t^2+t+1)(t^4+t^3+t^2+t+1)$. Moreover,

$$\widehat{P}_\Gamma(t) = \frac{t^8 - t^6 + t^4 - t^2 + 1}{t^{12} - 2\,t^{10} - t^9 + t^8 + t^7 + t^5 + t^4 - t^3 - 2\,t^2 + 1}.$$

Because of the to big size of the numerators $N_\Gamma(t,\, u,\, w)_i$'s, only the denominator is given in the text: all the numerators may be found on the web (`http://math.univ-lyon1.fr/~butin/`).

References

[BKR01] Bridgeland T., King A., Reid M., The McKay correspondence as an equivalence of derived categories, *J. Amer. Math. Soc.* **14** (2001), 535–554.

[BP09] F. Butin, G. S. Perets, McKay correspondence and the branching law for finite subgroups of $\mathbf{SL}_3\mathbb{C}$, accepted for publication in *Journal of Group Theory*, arXiv:0909.0578.

[GNS04] Gomi Y., Nakamura I., Shinoda K., Coinvariant Algebras of Finite Subgroups of $\mathbf{SL}_3\mathbb{C}$, *Canad. J. Math.* **56** (3), 495–528, 2004.

[GSV83] Gonzalez-Sprinberg G., Verdier J.-L., Construction géométrique de la correspondance de McKay, *Annales scientifiques de l'E. N. S., 4ème série* **16** (3), 409–449, 1983.

[HH01] A. Hanany, Y.-H. He, A monograph on the classification of the discrete subgroups of $\mathbf{SU}(4)$, *JHEP* **27**, 2001.

[Kos85] Kostant B., The McKay Correspondence, the Coxeter Element and Representation Theory, *SMF, Astérisque, hors série*, 209–255, 1985.

[Kos06] Kostant B., The Coxeter element and the branching law for the finite subgroups of $\mathbf{SU}(2)$, The Coxeter legacy, 63–70, *Amer. Math. Soc.*, Providence, RI, 2006, and arXiv:math/0411142v1 [math.RT], 2004.